# HOSSAM GHANEM

## (43) 5.3 *THE FUNDAMENTAL THEOREM OF CALCULUS (C)*

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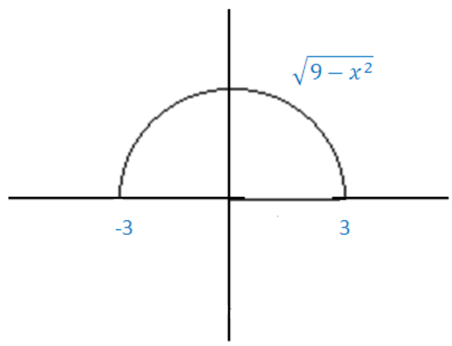
| **Example 1**<br>12 January 9, 1995 | Evaluate the following integral $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ |

### Solution

$$I = \int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{1}{2}\,\pi\,(3)^2 = \frac{9}{2}\,\pi$$

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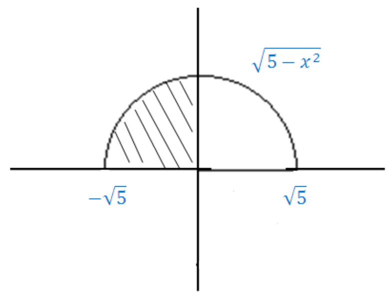
| **Example 2**<br>5 July 13, 1992 | Find $\int_{-\sqrt{5}}^{0} \sqrt{5-x^2}\,dx$ |

### Solution

$$I = \int_{-\sqrt{5}}^{0} \sqrt{5-x^2}\,dx = \frac{1}{4}\,\pi\,\left(\sqrt{5}\right)^2 = \frac{5}{4}\,\pi$$



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| **Example 3** | Evaluate the following integral $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^4 \tan\frac{x}{2}\,dx$ |

Solution

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^4 \tan\frac{x}{2}\,dx$$

Let $f(x) = x^4 \tan\frac{x}{2}$

$$f(-x) = (-x^4)\tan\frac{-x}{2} = -x^4\tan\frac{x}{2} = -f(x)$$

$\therefore f$ is odd function

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^4 \tan\frac{x}{2}\,dx = 0$$

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| **Example 4**<br>25 January 12 .2003 | Evaluate $\int_{-2}^{2}\left(\cos(\pi x) - \frac{x^3}{\sqrt{1+x^4}}\right)dx$ |

Solution

$$I = \int_{-2}^{2}\left(\cos(\pi x) - \frac{x^3}{\sqrt{1+x^4}}\right)dx$$

$$I_1 = \int_{-2}^{2}\cos(\pi x)\,dx$$

$f(x) = \cos(\pi x)$

$f(-x) = \cos(-\pi x) = \cos(-\pi x) = f(x)$

$\therefore f$ even function

$$I_1 = 2\int_{0}^{2}\cos(\pi x)\,dx = 2\cdot\frac{1}{\pi}\Big[\sin(\pi x)\Big]_0^2 = \frac{2}{\pi}(\sin 2\pi - \sin 0) = \frac{2}{\pi}(0-0) = 0$$

$$I_2 = \int_{-2}^{2}\left(\frac{x^3}{\sqrt{1+x^4}}\right)dx$$

$$g(x) = \frac{x^3}{\sqrt{1+x^4}}$$

$$g(-x) = \frac{-x^3}{\sqrt{1+x^4}} = -g(x)$$

$\therefore g$ is odd function

$I_2 = 0$

$I = I_1 - I_2 = 0 - 0 = 0$

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| **Example 5**<br>23 May 26. 2002 | Evaluate $\displaystyle\int_{-3}^{3}\left(\sqrt{9-x^2}+x^2\sin^3 x+x\right)dx$ |

## Solution

$$I=\int_{-3}^{3}\left(\sqrt{9-x^2}+x^2\sin^3 x+x\right)dx$$

$$I_1=\int_{-3}^{3}\left(\sqrt{9-x^2}\right)dx=\frac{1}{2}\pi(3)^2=\frac{9}{2}\pi$$

$$I_2=\int_{-3}^{3}(x^2\sin^3 x+x)\,dx$$

$f(x)=x^2\sin^3 x+x$

$f(-x)=x^2\sin^3(-x)-x=-x^2\sin^3 x-x=-(x^2\sin^3 x+x)=-f(x)$

$\therefore$ $f$ is odd function

$I_2=0$

$$I=I_1+I_2=\frac{9}{2}\pi$$

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| **Example 6**<br>40 August 7, 2011 | (3 Points ) Evaluate $\displaystyle\int_{-2}^{2}\left(\sqrt{4-x^2}+\sin(x^3)\right)dx$ |

## Solution

$$I=\int_{-2}^{2}\left(\sqrt{4-x^2}+\sin(x^3)\right)dx$$

$$I_1=\int_{-2}^{2}\left(\sqrt{4-x^2}\right)dx=\frac{1}{2}\pi(2)^2=\frac{4}{2}\pi=2\pi$$

$$I_2=\int_{-2}^{2}\sin(x^3)\,dx$$

$f(x)=\sin^3 x$

$f(-x)=\sin^3(-x)=-\sin^3 x=-f(x)$

$\therefore$ $f$ is odd function

$I_2=0$

$I=I_1+I_2=2\pi+0=2\pi$

| Example 7<br>33 January 20, 2009 | Evaluate $\int_{0}^{3} \sqrt{3-x}\, dx + \int_{-2}^{2} x^3 \cos x\, dx.$ |
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## Solution

$$I = \int_{0}^{3} \sqrt{3-x}\, dx + \int_{-2}^{2} x^3 \cos x\, dx.$$

$$I_1 = \int_{0}^{3} \sqrt{3-x}\, dx = \int_{0}^{3} (3-x)^{\frac{1}{2}}\, dx = \frac{-2}{3}\left[(3-x)^{\frac{3}{2}}\right]_0^3 = \frac{-2}{3}\left(0 - 3^{\frac{3}{2}}\right) = \frac{-2}{3}\left(-3\sqrt{3}\right) = 2\sqrt{3}$$

$$I_2 = \int_{-2}^{2} x^3 \cos x\, dx.$$

$f(x) = x^3 \cos x$

$f(-x) = -x^3 \cos(-x) = -x^3 \cos x = -f(x)$

$\therefore$ $f$ is odd function

$I_2 = 0$

$I = I_1 + I_2 = 2\sqrt{3} + 0 = 2\sqrt{3}$

| Example 8<br>19 July 29, 2000 | Find $f^{\backslash}(x)$ if $f(x) = \int_{x^3}^{\tan x} 3t^3\, dt$ |
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## Solution

$$f(x) = \int_{x^3}^{\tan x} 3t^3\, dt$$

$$f^{\backslash}(x) = 3(\tan x)^3 \cdot \sec^2 x - 3(x^3)^3(3x^2) = 3\tan^3 x \sec^2 x - 9x^{11}$$

| Example 9 | Evaluate $\int_{0}^{1} \left\{ D_x \int_{5}^{x^2} \left(3\sqrt{s} + 10s\right) ds \right\} dx$ |
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## Solution

$$I_1 = \int_{0}^{1} \left\{ D_x \int_{5}^{x^2} \left(3\sqrt{s} + 10s\right) ds \right\} dx = \int_{0}^{1} \left(\left(3x + 10(x^2)\right) \cdot 2x\right) dx = \int_{0}^{1} \left(6x^2 + 20x^3\right) dx$$

$$= \left[2x^3 + 5x^4\right]_0^1 = 2 + 5 - 0 = 7$$

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| Example 10<br>12 January 9, 1995 | Show that the function $f(x) = \int_0^{\frac{1}{x}} \frac{dt}{t^2+1} + \int_0^{x} \frac{dt}{t^2+1}$ is constant for $x > 0$ |

## Solution

$$f(x) = \int_0^{\frac{1}{x}} \frac{dt}{t^2+1} + \int_0^{x} \frac{dt}{t^2+1}$$

$$f^{\backslash}(x) = \frac{1}{\frac{1}{x^2}+1} \cdot \frac{-1}{x^2} + \frac{1}{x^2+1} = \frac{-1}{1+x^2} + \frac{1}{x^2+1} = 0$$

$$f^{\backslash}(x) = 0$$

$$f(x) = c$$

$\therefore\ f$ is constant for $x > 0$

# Homework

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| 1 | Evaluate | $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ | |
| 2 | Evaluate | $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ | 18 May 24 ,2000 |
| 3 | Evaluate | $\int_{0}^{\sqrt{3}} \sqrt{3-x^2}\,dx$ | |
| 4 | Evaluate | $\int_{-\pi}^{\pi} x^2 \sin 2x\,dx$ | 13 February 19, 1995 |
| 5 | Evaluate | $\int_{-1}^{1} \left(t^3 + 2\sqrt{1-t^2}\right) dt.$ | 22 August 11.2001 A |
| 6 | Evaluate | $\int_{0}^{1} \left(x + 2\sqrt{1-x^2}\right) dx.$ | 26 June 7, 2003 |
| 7 | Evaluate | $\int_{-1}^{0} \left\{ D_x \int_{5}^{x^2} (2\sqrt{s} - 5s)\,ds \right\} dx$ | |
| 8 | Find $f^{\backslash}(x)$ if | $f(x) = \int_{3x}^{x^2} \sqrt{\sin^2 t + 7t^6}\,dt$ | 20 January 3 ,2001 |
| 9 | Evaluate | $\int_{-\pi}^{\pi} \left(x^3 + \sqrt{\pi^2 - x^2}\right) dx$ | 30 Jan. 12. 2008 |
| 10 | Let | $f(x) = D_x\left(\int_{\cos x}^{\sin x} 3t^2\,dt\right)$ Find $f\left(\frac{\pi}{4}\right)$ | |
| 11 | Find | $\frac{d}{dx}\left(\int_{x}^{x^2} \frac{dt}{1+t^3}\right)$ | |

# Homework

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| 12 | Let $f(x) = D_x\left(\int_{\cos x}^{\sin x} t^3\,dt\right)$ Find $f\left(\frac{\pi}{4}\right)$ |
| 13 | Find $\frac{d}{dx}\left(\int_{x}^{x^3} \frac{dt}{1+t^2}\right)$ |
| 14 | Find the second derivative with respect to $x$ of function $y = \int_0^{x^3} \frac{dt}{(t^3+1)}$ at $x = 1$ — 13 February 19, 1995 |
| 15 | Show that the following function is constant on $(0, \infty)$ : $F(x) = \int_x^{3x} \frac{1}{t}\,dt$ — 25 January 12 .2003 |
| 16 | Let $F(x) = 2\int_1^{3x} \frac{1}{t}\,dt - \int_2^{x^2} \frac{1}{t}\,dt$ show that $F$ is an constant function on $[1, \infty)$ — 27 May 30. 2006 |
| 16 | Let $F(x) = 2\int_1^{3x} \frac{1}{t}\,dt - \int_2^{x^2} \frac{1}{t}\,dt$ show that $F$ is an constant function on $[1, \infty)$ — 27 May 30. 2006 |
| 17 | [4 Pts. ] Find an equation of the tangent line to the graph of $f(x) = \int_2^{3x-x^2} \frac{1}{t^2+4}\,dt$ — 41 7 January 2012 |